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Asymptotic Formulas for Convolution Operators with Spline Kernels

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We derive asymptotic formulas for convolution operators with spline kernels for differentiable functions. These formulas are analogous to Bernstein's extension of Voronovskaya's results on Bernstein polynomials for functions with higher order derivatives. Two classes of operators are considered, viz., the de la Vallée Poussin–Schoenberg operators with trigonometric spline kernels and the singular integrals of Riemann–Lebesgue with periodic polynomial spline kernels. The former includes the de la Vallée means as a special case. © 1995 Academic Press, Inc.

1. INTRODUCTION

The variation diminishing spline operators or Bernstein–Schoenberg operators for continuous functions are a spline extension of the Bernstein polynomial operators. Their properties are reminiscent of those of the Bernstein polynomials. They were introduced by Schoenberg [15], where among other results an analogue of Voronovskaya's formula on the asymptotic behaviour of the operators for twice differentiable functions was stated. Recently Marsden and Riemenschneider [11] (see also [7]) gave an extension of the asymptotic formula for functions with higher order

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derivatives; an extension which was in line with Bernstein's extension of Voronovskaya's result for the Bernstein polynomial operators [1].

The de la Vallée Poussin means of a 2π-periodic integrable function f,

$$V_m(f; x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \omega_m(x-t)\, dt, \qquad x \in [0, 2\pi), \tag{1.1}$$

where

$$\omega_m(x) := \sum_{\nu=-m}^{m} \frac{(m!)^2}{(m-\nu)!\,(m+\nu)!} e^{i\nu x}, \qquad x \in \mathbf{R}, \tag{1.2}$$

and m a positive integer, are trigonometric analogues of the Bernstein polynomials. They are shape-preserving trigonometric convolution operators [13]. A spline extension of the de la Vallée Poussin means consists of the convolution operators

$$T_m(f; x) \equiv T_{m,k}(f; x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \tau_{m,k}(x-t)\, dt, \tag{1.3}$$

where m, k are positive integers with $k \geqslant 2m+1$ and

$$\tau(x) \equiv \tau_{m,k}(x) := \sum_{\nu \in Z} \hat{\tau}_\nu e^{i\nu x} \tag{1.4}$$

with Fourier coefficients

$$\hat{\tau}_\nu = \begin{cases} \dfrac{(m!)^2 (\sin(m-\nu)\, h/2 \cdots \sin h/2)(\sin(m+\nu)\, h/2 \cdots \sin h/2)}{(m-\nu)!\,(m+\nu)!\,(\sin h/2 \cdots \sin mh/2)^2}, & |\nu| \leqslant m \\[2ex] \dfrac{k(m!)^2 \sin(\nu-m)\, h/2 \sin(\nu-m+1)\, h/2 \cdots \sin(\nu+m)\, h/2}{\pi(\nu-m) \cdots (\nu+m)(\sin h/2 \cdots \sin mh/2)^2}, & |\nu| > m, \end{cases} \tag{1.5}$$

where $h := 2\pi/k$. The function $\tau_{m,k}$ is a trigonometric B-spline of degree m [16, 5]. We shall call T_m the *de la Vallée Poussin–Schoenberg operators*. If $k = 2m+1$, they reduce to the de la Vallée Poussin means.

A related sequence of operators is the sequence of *singular integrals of Riemann–Lebesgue* (see [2, p. 54]),

$$R_n(f; x) \equiv R_{n,k}(f; x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, b_{n,k}(x-t)\, dt, \tag{1.6}$$

which are convolution operators in which the kernels are defined by their Fourier series expansions

$$b_{n,k}(x) := \sum_{\nu \in Z} \left(\frac{\sin h\nu/2}{h\nu/2}\right)^n e^{i\nu x}, \qquad x \in \mathbf{R}, \tag{1.7}$$

where n, k are positive integers and $h := 2\pi/k$. The functions $b_{n,k}$ are the periodic polynomial B-splines of degree $n-1$ supported on the interval $[-nh/2, nh/2]$ (see [12] and also Section 4).

The following asymptotic formula for the de la Vallée Poussin means of a twice differentiable function is due to Natanson (see [8, p. 115]).

THEOREM 1.1 (Natanson). *If $f^{(2)}(x)$ exists,*

$$\lim_{m \to \infty} (m+1)\{V_m(f; x) - f(x)\} = f^{(2)}(x). \tag{1.8}$$

This is a trigonometric analogue of Voronovskaya's estimate for Bernstein polynomials. In line with Schoenberg's extension of Voronovskaya's theorem to variation diminishing spline operators, it was shown in [6] that the following holds for the de la Vallée Poussin–Schoenberg operators $T_{m,k}(f; \cdot)$ if $f^{(2)}(x)$ exists,

$$\lim_{\substack{m \to \infty \\ mh \to \alpha}} (m+1)\{T_{m,k}(f; x) - f(x)\} = \left(1 - \frac{\alpha}{2}\cot\frac{\alpha}{2}\right) f^{(2)}(x), \tag{1.9}$$

where the limit is taken as $m \to \infty$ and $mh \to \alpha \in (0, \pi]$.

Our object is to derive asymptotic formulas for the de la Vallée Poussin–Schoenberg operators and the singular integrals of Riemann–Lebesgue for higher order differentiable functions, in the same vein as Bernstein's extension of Voronovskaya's estimate for Bernstein polynomials, and Marsden and Riemenschneider's extension of Schoenberg's result on Bernstein–Schoenberg operators. The main theorems are stated in Section 2. A preliminary result on the asymptotic behaviour of a sequence of positive convolution operators with even kernels is given in Section 3. It depends on the asymptotic estimate for the trigonometric moments of their kernels. A detailed analysis of the asymptotic behaviour of the trigonometric moments of the periodic polynomial B-splines $b_{n,k}$, together with the proofs of Theorems 2.1, 2.2, and 2.3 are given in Section 4. The proof of Theorem 2.4 for the de la Vallée Poussin–Schoenberg operators is analogous to that of Theorem 2.1. Although the computations are more involved the analysis of the asymptotic behaviour of the trigonometric moments of $\tau_{m,k}$ is the same as that for $b_{n,k}$, and the details will be omitted. However, the same method does not work for Theorem 2.5. In this case an asymptotic estimate for the trigonometric moments of the trigonometric B-spline kernels is obtained via their recurrence relation. This is given in Section 5.

2. The Main Theorems

To state the main theorems we require the combinatorial numbers which are the coefficients in the expansion of the central factorial polynomials,

$$x^{[n]} := \begin{cases} x \prod_{j=1}^{n-1} \left(x - \frac{n}{2} + j\right), & n > 0 \\ 1, & n = 0, \end{cases} \tag{2.1}$$

where $n > 0$ is the degree of the polynomial $x^{[n]}$. The coefficients $t(n, j)$ in the expansion

$$x^{[n]} = \sum_{j=0}^{n} t(n, j)\, x^j, \qquad n \in \mathbf{N}_0, \tag{2.2}$$

are called the *central factorial numbers of the first kind* (see [14, p. 213]). In (2.2), $\mathbf{N}_0$ denotes the set of nonnegative integers. We shall also use $\mathbf{N}$ for the set of natural numbers.

The asymptotic formulas for the convolution operators involve the trigonometric moments of their kernels. For an even 2π-periodic integrable function ϕ, its *trigonometric moment of order* $2j$, $j \in \mathbf{N}_0$, is defined by

$$M_{2j}(\phi) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(2 \sin \frac{1}{2} t\right)^{2j} \phi(t)\, dt. \tag{2.3}$$

For s, $\nu \in \mathbf{N}$, let

$$a_{s,\nu}(\phi) := \sum_{j=\nu}^{s} \frac{(-1)^{j+\nu}\, t(2j, 2\nu)}{(2j)!} M_{2j}(\phi),$$

and let $C_{2\pi}$ denote the class of continuous 2π-periodic functions. We shall prove the following theorems.

THEOREM 2.1. *Suppose* $n, s \in \mathbf{N}$ *with* $s < n$. *If* $f \in C_{2\pi}$ *and its derivatives up to order* $2s$ *exist at* $x \in (-\pi, \pi)$, *then*

$$\lim_{h \to 0} \frac{1}{h^{2s}} \left\{ R_n(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = (-1)^s \beta_s^n f^{(2s)}(x), \tag{2.4}$$

where β_s^n *is a polynomial in n of degree s with leading coefficient* $(-1)^s/(4!)^s\, s!$. *Further,* β_s^n *can be evaluated by the following algorithm:*

For $\kappa \in \mathbf{N}$,

$$\beta_\kappa^1 := \frac{(-1)^\kappa}{(2\kappa+1)!\, 2^{2\kappa}} \tag{2.5}$$

and for $r = 2, 3, ..., n$,

$$\beta_\kappa^r := \sum_{\nu=0}^{\kappa} \beta_{\kappa-\nu}^{r-1} \beta_\nu^1. \tag{2.6}$$

THEOREM 2.2. *Suppose* $s \in \mathbf{N}$. *If* $f \in C_{2\pi}$ *and its derivatives up to order 2s exist at* $x \in (-\pi, \pi)$, *then*

$$\lim_{\substack{n \to \infty \\ nh \to 0}} \frac{1}{(nh^2)^s} \left\{ R_n(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = \left(\frac{1}{4!}\right)^s \frac{f^{(2s)}(x)}{s!}. \tag{2.7}$$

THEOREM 2.3. *If* $f \in C_{2\pi}$ *and its derivatives up to order 2s exist at* $x \in (-\pi, \pi)$, *then*

$$\lim_{\substack{n \to \infty \\ nh \to \beta}} n^s \left\{ R_n(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(b_{n,k}) f^{(2\nu)}(x) \right\} = \left(\frac{\beta^2}{4!}\right)^s \frac{f^{(2s)}(x)}{s!}, \tag{2.8}$$

where the limit is taken as $n \to \infty$ *and* $nh \to \beta > 0$.

Corresponding to Theorems 2.1 and 2.3 we have the following results for the de la Vallée Poussin–Schoenberg operators $T_{m,k}$. However, we are unable to obtain a similar result for $T_{m,k}$ corresponding to Theorem 2.2.

THEOREM 2.4. *Suppose* $m, s \in \mathbf{N}$ *with* $s < m$. *If* $f \in C_{2\pi}$ *and its derivatives up to order 2s exist at* $x \in (-\pi, \pi)$, *then*

$$\lim_{h \to 0} \frac{1}{h^{2s}} \left\{ T_{m,k}(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(\tau_{m,k}) f^{(2\nu)}(x) \right\} = (-1)^s \alpha_s^m f^{(2s)}(x), \tag{2.9}$$

where α_s^m *is a polynomial in m of degree s with leading coefficient* $(-1)^s/(3!\,2)^s\, s!$. *Further,* α_s^m *can be evaluated by the following algorithm:*

For $\kappa \in \mathbf{N}$,

$$\alpha_\kappa^0 := \frac{(-1)^\kappa}{(2\kappa+1)!\, 2^{2\kappa}}, \tag{2.10}$$

and for $r = 1, \ldots, m$,

$$\alpha_\kappa^r := \sum_{\nu=0}^{\kappa} \frac{2^{2\nu}\alpha_{\kappa-\nu}^{r-1}\alpha_\nu^0}{\nu+1}. \tag{2.11}$$

THEOREM 2.5. *If* $f \in C_{2\pi}$ *and its derivatives up to order* $2s$ *exist at* $x \in (-\pi, \pi)$,

$$\lim_{\substack{m\to\infty \\ mh\to\alpha}} m^s \left\{ T_{m,k}(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(\tau_{m,k}) f^{(2\nu)}(x) \right\} = \left(1 - \frac{\alpha}{2}\cot\frac{\alpha}{2}\right)^s \frac{f^{(2s)}(x)}{s!}, \tag{2.12}$$

where the limit is taken as $m \to \infty$ *and* $mh \to \alpha \in (0, \pi]$.

The special case of Theorem 2.5 with $k = 2m+1$ or $h = 2\pi/(2m+1)$ gives the following result on the asymptotic estimate for the de la Vallée Poussin means $V_m(f; x)$ for functions with higher order derivatives. In this case $T_{m,k} = V_m$ and $\alpha = \pi$.

COROLLARY 2.1. *If* $f \in C_{2\pi}$ *and its derivatives up to order* $2s$ *exist at* $x \in (-\pi, \pi)$,

$$\lim_{m\to\infty} m^s \left\{ V_m(f; x) - \sum_{\nu=0}^{s-1} a_{s,\nu}(\omega_m) f^{(2\nu)}(x) \right\} = \frac{f^{(2s)}(x)}{s!}. \tag{2.13}$$

3. POSITIVE CONVOLUTION OPERATORS WITH EVEN KERNELS

For $n \in \mathbf{N}$, let

$$K_n(f; x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, k_n(x-t), \qquad f \in C_{2\pi}, \tag{3.1}$$

be a sequence of positive convolution operators with even kernels k_n which are nonnegative and normalized so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t)\, dt = 1. \tag{3.2}$$

The asymptotic formulas for K_n involve the trigonometric moments of its kernel k_n. We shall require the following Taylor expansion (see [4, 19]),

$$(\arcsin x)^p = \sum_{j=0}^{\infty} (-1)^j 2^{2j} \frac{p!}{(p+2j)!} t(p+2j, p)\, x^{p+2j}, \qquad |x| < 1, \tag{3.3}$$

where $p \in \mathbf{N}$. For even p, (3.3) can be written in the form

$$t^{2v} = \sum_{j=v}^{\infty} (-1)^{j+v} \frac{(2v)!}{(2j)!} t(2j, 2v) \left(2 \sin \frac{1}{2} t\right)^{2j}, \qquad |t| < \pi, \tag{3.4}$$

where $t(n, j)$ are the central factorial numbers.

We observe that for $j \in \mathbf{N}$, $x^{[2j]} = \prod_{l=0}^{j-1} (x^2 - l^2)$. Therefore,

$$\sum_{v=0}^{2j} t(2j, v)\, x^v = \prod_{l=0}^{j-1} (x^2 - l^2). \tag{3.5}$$

It follows that $t(2j, 2j) = 1$ and $t(2j, 0) = t(2j, 2v-1) = 0$, for all v. Furthermore, $t(2m, 2v)$ satisfy the following partial difference equation:

$$t(2m+2, 2v) = t(2m, 2v-2) - m^2 t(2m, 2v), \tag{3.6}$$

with initial conditions

$$t(2, 0) = 0, \qquad t(2, 2) = 1. \tag{3.7}$$

In Eq. (3.6) which is readily obtained from (3.5), we have assumed that $t(2m, 2v) = 0$ for $v < 0$ or $v > m$. It follows easily from (3.5) that

$$\operatorname{sgn}(t(2j, 2v)) = (-1)^{j+v}, \qquad v = 1, 2, ..., j.$$

Hence the series in (3.4) is a positive series.

THEOREM 3.1. *Suppose for* $j \in \mathbf{N}$, *the limit* $\lim_{n \to \infty} n^j M_{2j}(k_n)$ *exists and*

$$\lim_{n \to \infty} n^j M_{2j}(k_n) = \lambda_j. \tag{3.8}$$

If $f \in C_{2\pi}$ *and its derivatives up to order* $2s$ *exist at* $x \in (-\pi, \pi)$, *then*

$$\lim_{n \to \infty} n^s \left\{ K_n(f; x) - \sum_{v=0}^{s-1} a_{s,v}(k_n) f^{(2v)}(x) \right\} = \lambda_s \frac{f^{(2s)}(x)}{(2s)!}. \tag{3.9}$$

Proof. For $t \in (-\pi, \pi)$, Taylor's formula about x gives

$$f(x+t) = \sum_{v=0}^{s} \frac{f^{(2v)}(x)}{(2v)!} t^{2v} + \sum_{v=0}^{s-1} \frac{f^{(2v+1)}(x)}{(2v+1)!} t^{2v+1} + g(t)\, t^{2s}, \tag{3.10}$$

where g is continuous and $\lim_{t\to 0} g(t)=0$. Using (3.4) one can express

$$f(x+t)=\sum_{\nu=0}^{s} f^{(2\nu)}(x)\sum_{j=\nu}^{s}(-1)^{j+\nu}\frac{t(2j,2\nu)}{(2j)!}\left(2\sin\frac{1}{2}t\right)^{2j}$$
$$+\sum_{\nu=0}^{s} f^{(2\nu)}(x)\sum_{j=s+1}^{\infty}(-1)^{j+\nu}\frac{t(2j,2\nu)}{(2j)!}\left(2\sin\frac{1}{2}t\right)^{2j}$$
$$+\sum_{\nu=0}^{s-1}\frac{f^{(2\nu+1)}(x)}{(2\nu+1)!}t^{2\nu+1}+g(t)\,t^{2s}. \tag{3.11}$$

Since k_n is even, (3.11) leads to

$$K_n(f;x)=\sum_{\nu=0}^{s-1} a_{s,\nu}(k_n)f^{(2\nu)}(x)+M_{2s}(k_n)\frac{f^{(2s)}(x)}{(2s)!}+S_{1,n}+S_{2,n}, \tag{3.12}$$

where

$$S_{1,n}:=\sum_{\nu=0}^{s} f^{(2\nu)}(x)\sum_{j=s+1}^{\infty}(-1)^{j+\nu}\frac{t(2j,2\nu)}{(2j)!}M_{2j}(k_n), \tag{3.13}$$

$$S_{2,n}:=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,k_n(t)\,t^{2s}\,dt. \tag{3.14}$$

Therefore,

$$\lim_{n\to\infty} n^s\left\{K_n(f;x)-\sum_{\nu=0}^{s-1} a_{s,\nu}(k_n)f^{(2\nu)}(x)\right\}$$
$$=\lambda_s\frac{f^{(2s)}(x)}{(2s)!}+\lim_{n\to\infty} n^s S_{1,n}+\lim_{n\to\infty} n^s S_{2,n}. \tag{3.15}$$

The first limit on the right of (3.15) is zero. To prove this, we observe that

$$0\leqslant\frac{n^s}{2\pi}\int_{1/\sqrt{n}\leqslant|t|\leqslant\pi}\left(2\sin\frac{t}{2}\right)^{2j}k_n(t)\,dt\leqslant n^s M_{2j}(k_n).$$

Hence (3.8) implies that

$$\lim_{n\to\infty}\frac{n^s}{2\pi}\int_{1/\sqrt{n}\leqslant|t|\leqslant\pi}\left(2\sin\frac{t}{2}\right)^{2j}k_n(t)\,dt=0,\qquad s=0,1,...,j-1.$$

For any $\varepsilon>0$, choose N so that

$$\frac{n^s}{2\pi}\int_{1/\sqrt{n}\leqslant|t|\leqslant\pi}\left(2\sin\frac{t}{2}\right)^{2j}k_n(t)\,dt<\varepsilon,\qquad n\geqslant N.$$

Then

$$n^s M_{2j}(k_n) \leqslant \frac{n^s}{2\pi} \int_{|t| < 1/\sqrt{n}} \left(2 \sin \frac{t}{2}\right)^{2j} k_n(t)\, dt + \varepsilon$$

$$\leqslant \frac{1}{n^{j-s}} + \varepsilon.$$

It follows from (3.13) that

$$|n^s S_{1,n}| \leqslant C \sum_{j=s+1}^{\infty} (-1)^{j+v} \frac{t(2j, 2v)}{(2j)!} \left(\frac{1}{n}\right)^{j-s}$$

$$+ C\varepsilon \sum_{j=s+1}^{\infty} (-1)^{j+v} \frac{t(2j, 2v)}{(2j)!},$$

where C is a constant independent of n. Since the series

$$\sum_{j=s+1}^{\infty} (-1)^{j+v} \frac{t(2j, 2v)}{(2j)!}$$

converges absolutely and ε is arbitrary, it follows that

$$\lim_{n \to \infty} n^s S_{1,n} = 0.$$

To show that the second limit is also zero, let $\varepsilon > 0$. Choose $\delta > 0$ such that $|g(t)| < \varepsilon$ whenever $|t| < \delta$, and write

$$n^s S_{2,n} = I_1 + I_2, \tag{3.16}$$

where

$$I_1 := \frac{n^s}{2\pi} \int_{|t| < \delta} g(t)\, t^{2s} k_n(t)\, dt,$$

$$I_2 := \frac{n^s}{2\pi} \int_{\delta \leqslant |t| \leqslant \pi} g(t)\, t^{2s} k_n(t)\, dt.$$

Because of the inequality $t \leqslant \pi \sin \frac{1}{2} t$, $t \in [0, \pi]$,

$$|I_1| \leqslant \frac{\varepsilon}{2\pi} \left(\frac{\pi}{2}\right)^{2s} n^s M_{2s}(k_n) \tag{3.17}$$

which is arbitrarily small, since $n^s M_{2s}(k_n)$ is bounded. On the other hand,

$$|I_2| \leqslant \frac{n^s \|g\|}{2\pi} \int_{\delta \leqslant |t| \leqslant \pi} \left(\frac{t}{\delta}\right)^2 t^{2s} k_n(t)\, dt$$

$$\leqslant \frac{n^s \|g\|}{2\pi\delta^2} \left(\frac{\pi}{2}\right)^{2(s+1)} M_{2(s+1)}(k_n) \tag{3.18}$$

which tends to zero as $n \to \infty$, by (3.8). Hence $\lim_{n\to\infty} n^s S_{2,n} = 0$. ▮

In order to establish the main theorems we need only to study the asymptotic behaviour of the trigonometric moments of the periodic polynomial B-splines $b_{n,k}$ and the trigonometric B-splines $\tau_{m,k}$.

4. Trigonometric Moments of Periodic Polynomial B-Splines

Let $M_1 := \chi_{(-1/2,1/2]}$ be the characteristic function of the interval $(-\frac{1}{2}, \frac{1}{2}]$ and for $n = 2, 3, \ldots$, let $M_n := M_1 * M_{n-1}$ be the uniform polynomial B-spline of degree $n-1$ (see [21, 22]). Here $*$ denotes the operation of convolution. The Fourier transform of M_n is

$$\hat{M}_n(u) = \left(\frac{\sin u/2}{u/2}\right)^n.$$

Let k be a positive integer, $h := 2\pi/k$ and for $n = 1, 2, \ldots$, we define the uniform 2π-periodic polynomial B-spline of degree $n-1$ by

$$b_{n,k}(x) := \sum_{v\in\mathbf{Z}} k M_n(h^{-1}(x - 2\pi v)), \qquad x \in \mathbf{R}. \tag{4.1}$$

The Fourier coefficients of $b_{n,k}$ can be computed from the Fourier transform of M_n, viz.,

$$\hat{b}_{n,k}(v) = \hat{M}_n(hv) = \left(\frac{\sin hv/2}{hv/2}\right)^n, \qquad v \in \mathbf{Z}. \tag{4.2}$$

Therefore,

$$b_{n,k}(x) = \sum_{v\in\mathbf{Z}} \hat{b}_{n,k}(v)\, e^{ivx}. \tag{4.3}$$

The function $b_{n,k}(x)$ is even, positive, and 2π-periodic. Further, $\hat{b}_{n,k}(0) = 1$ and $\hat{b}_{n,k}(v) \to 1$ as $k \to \infty$ (i.e., $h \to 0$) for all $v \in \mathbf{Z}$.

Let $\sigma \in \mathbf{N}$. For an even 2π-periodic integrable function k_n, its trigonometric moment of order 2σ can be expressed as

$$M_{2\sigma}(k_n) = (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j \hat{k}_n(\sigma - j). \tag{4.4}$$

The relation (4.4) is obtained from the definition (2.3) by expressing $2 \sin \frac{1}{2} t$ in terms of exponentials and then expanding the resulting expression by binomial expansion. By (4.4), the trigonometric moments of the periodic B-spline kernel $b_{n,k}$ are given by

$$M_{2\sigma}(b_{n,k}) = (-1)^\sigma \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^j B(n, j; h), \tag{4.5}$$

where

$$B(l, j; h) := \left(\frac{\sin((\sigma - j) h/2)}{(\sigma - j) h/2} \right)^l, \qquad l = 1, 2, ..., n; \quad j = 0, 1, ..., 2\sigma. \tag{4.6}$$

For $l = 1, 2, ..., n$ and $j = 0, 1, ..., 2\sigma$, we define a sequence $(B_{2\kappa}(l, j))_{\kappa \in \mathbf{N}_0}$ by

$$B(l, j; h) =: \sum_{\kappa=0}^{\infty} B_{2\kappa}(l, j) h^{2\kappa}. \tag{4.7}$$

LEMMA 4.1. *For $l = 1, 2, ..., n$ and $j = 0, 1, ..., 2\sigma$,*

$$B_{2\kappa}(l, j) = \beta_\kappa^l (\sigma - j)^{2\kappa}, \qquad \kappa \in \mathbf{N}_0, \tag{4.8}$$

where

$$\beta_\kappa^1 = \frac{(-1)^\kappa}{(2\kappa + 1)! \, 2^{2\kappa}}, \tag{4.9}$$

and for $l = 2, 3, ..., n$,

$$\beta_\kappa^l = \sum_{\nu=0}^{\kappa} \beta_{\kappa-\nu}^{l-1} \beta_\nu^1. \tag{4.10}$$

Proof. For $l = 1$, we have

$$B(1, j; h) = \frac{\sin((\sigma - j) h/2)}{(\sigma - j) h/2}.$$

If $j \neq \sigma$, expanding the sine function in powes of h gives

$$\sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} (\sigma-j)^{2\kappa} h^{2\kappa}}{(2\kappa+1)!\, 2^{2\kappa}} = \sum_{\kappa=0}^{\infty} B_{2\kappa}(1, j)\, h^{2\kappa}.$$

It follows that

$$B_{2\kappa}(1, j) = \frac{(-1)^{\kappa} (\sigma-j)^{2\kappa}}{(2\kappa+1)!\, 2^{2\kappa}} = \beta_{\kappa}^{1}(\sigma-j)^{2\kappa}, \tag{4.11}$$

where

$$\beta_{\kappa}^{1} = \frac{(-1)^{\kappa}}{2^{2\kappa}(2\kappa+1)!}.$$

Note that for $j=\sigma$, $B(1, \sigma; h) = 1$, and again (4.11) holds.

Suppose that for $l < n$,

$$B_{2\kappa}(l, j) = \beta_{\kappa}^{l}(\sigma-j)^{2\kappa}. \tag{4.12}$$

From (4.6),

$$B(l+1, j; h) = B(l, j; h) \frac{\sin((\sigma-j)\, h/2)}{(\sigma-j)\, h/2}.$$

Expanding the second factor in powers of h and using (4.7) leads to

$$B(l+1, j; h) = \left(\sum_{\kappa=0}^{\infty} B_{2\kappa}(l, j)\, h^{2\kappa} \right) \left(\sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} (\sigma-j)^{2\kappa} h^{2\kappa}}{(2\kappa+1)!\, 2^{2\kappa}} \right).$$

Taking the Cauchy product gives

$$B(l+1, j; h) = \sum_{\kappa=0}^{\infty} \left(\sum_{\nu=0}^{\kappa} B_{2(\kappa-\nu)}(l, j) \frac{(-1)^{\nu} (\sigma-j)^{2\nu}}{(2\nu+1)!\, 2^{2\nu}} \right) h^{2\kappa}.$$

It follows from (4.7) that

$$B_{2\kappa}(l+1, j) = \sum_{\nu=0}^{\kappa} B_{2(\kappa-\nu)}(l, j) \frac{(-1)^{\nu} (\sigma-j)^{2\nu}}{(2\nu+1)!\, 2^{2\nu}},$$

and, using (4.12), a straightforward computation leads to

$$B_{2\kappa}(l+1, j) = \left(\sum_{\nu=0}^{\kappa} \beta_{\kappa-\nu}^{l} \beta_{\nu}^{1} \right) (\sigma-j)^{2\kappa} = \beta_{\kappa}^{l+1}(\sigma-j)^{2\kappa},$$

where $\beta_{\kappa}^{l+1} = \sum_{\nu=0}^{\kappa} \beta_{\kappa-\nu}^{l} \beta_{\nu}^{1}$. ∎

LEMMA 4.2. *Let $j \in \mathbf{N}$, and for any $r \in \mathbf{N}$ let*

$$S_j(r) := \sum_{\nu=1}^{r} \nu^{j-1}.$$

Then $S_j(r)$ is a polynomial in r of degree j with leading coefficient $1/j$.

Proof. For $r \geqslant 1$,

$$S_j(r) - S_j(r-1) = r^{j-1}$$

which is a difference equation for which the solution is of the form

$$S_j(r) = \sum_{\nu=0}^{j} a_\nu r^\nu,$$

where a_ν are constants. Therefore,

$$\begin{aligned} r^{j-1} = S_j(r) - S_j(r-1) &= \sum_{\nu=1}^{j} a_\nu \sum_{l=0}^{\nu-1} (-1)^{\nu-1-l} \binom{\nu}{l} r^l \\ &= \sum_{l=0}^{j-1} \left\{ \sum_{\nu=l+1}^{j} (-1)^{\nu-1} a_\nu \binom{\nu}{l} \right\} (-1)^l r^l. \end{aligned}$$

Equating the coefficient of r^{j-1} gives $a_j = 1/j$. ∎

LEMMA 4.3. *Let $\kappa \in \mathbf{N}_0$. For $l = 1, 2, ..., n$,*

$$\beta_\kappa^l = \frac{(-1)^\kappa l^\kappa}{(4!)^\kappa \kappa!} + \text{polynomial in } l \text{ of degree } < \kappa. \tag{4.13}$$

Proof. We shall establish the result by induction on κ using (4.9) and (4.10). For $\kappa = 0$, $B_0(l, j) = 1$. Hence $\beta_0^l = 1$ for all l. By (4.10),

$$\beta_1^l = \beta_1^{l-1} - \frac{1}{3!\, 2^2} \tag{4.14}$$

for $l \geqslant 2$. Repeated application of (4.14) gives

$$\beta_1^l = \beta_1^1 - \frac{(l-1)}{3!\, 2^2} = \frac{(-1)\, l}{4!}$$

for any $l \geqslant 1$, by (4.9). Hence (4.13) holds for $\kappa = 1$. Suppose it holds for $\kappa < s$ and for all $l \geqslant 1$. Using (4.10) and noting that $\beta_0^1 = 1$, we have

$$\beta_s^j - \beta_s^{j-1} = \sum_{\nu=1}^{s} \beta_{s-\nu}^{j-1} \beta_\nu^1 \tag{4.15}$$

for any integer $j \geqslant 2$. Summing (4.15) for j from 2 to l leads to

$$\beta_s^l - \beta_s^1 = \sum_{v=1}^{s} \sum_{j=2}^{l} \beta_{s-v}^{j-1} \frac{(-1)^v}{(2v+1)!\, 2^{2v}}. \tag{4.16}$$

Applying the inductive hypothesis to the summand on the right of (4.16) leads to

$$\begin{aligned} \beta_s^l - \beta_s^1 &= \sum_{j=2}^{l} \left(\beta_{s-1}^{j-1} \frac{(-1)}{4!} + \beta_{s-2}^{j-1} \frac{1}{5!\, 2^2} + \cdots + \frac{\beta_0^{j-1}(-1)^s}{(2s+1)!\, 2^{2s}} \right) \\ &= \frac{(-1)^s}{(4!)^s (s-1)!} \sum_{j=2}^{l} \{(j-1)^{s-1} \\ &\qquad + \text{polynomial in } (j-1) \text{ of degree } < s-1\}. \end{aligned}$$

By Lemma 4.2, the leading term in $\sum_{j=2}^{l} (j-1)^{s-1}$ is l^s/s. It follows from (4.17) that

$$\beta_s^l = \frac{(-1)^s\, l^s}{(4!)^s\, s!} + \text{polynomial in } l \text{ of degree } < s. \quad \blacksquare$$

LEMMA 4.4. *For $\kappa \in \mathbf{N}$ and $l = 1, 2, \ldots,$*

$$|\beta_\kappa^l| \leqslant \frac{l^\kappa}{\kappa!\, 2^\kappa}. \tag{4.18}$$

Proof. By (4.9), it is clear that the inequality (4.18) holds for $l = 1$ and for all $\kappa \in \mathbf{N}_0$. Suppose that it holds for $l = n$ and for all $\kappa \in \mathbf{N}_0$. Then by (4.10) and the inductive hypothesis,

$$\begin{aligned} |\beta_\kappa^{n+1}| &\leqslant \sum_{v=0}^{\kappa} |\beta_{\kappa-v}^n| |\beta_v^1| \\ &\leqslant \frac{n^\kappa}{\kappa!\, 2^\kappa} \sum_{v=0}^{\kappa} \binom{\kappa}{v} \left(\frac{1}{n}\right)^v. \end{aligned}$$

A straightforward simplification of the expression on the right of the last inequality leads to

$$|\beta_\kappa^{n+1}| \leqslant \frac{(n+1)^\kappa}{\kappa!\, 2^\kappa}, \qquad \kappa \in \mathbf{N}_0.$$

The result now follows by induction. $\blacksquare$

LEMMA 4.5. *For any* $\sigma \in \mathbf{N}$,

$$M_{2\sigma}(b_{n,k}) = (-1)^{\sigma}(2\sigma)!\,\beta_{\sigma}^{n} h^{2\sigma} + O(n^{\sigma+1}h^{2\sigma+2}) \qquad \text{as} \quad nh^2 \to 0. \tag{4.19}$$

Proof. Using (4.5), (4.7), and (4.8), we have

$$\begin{aligned} M_{2\sigma}(b_{n,k}) &= \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^{\sigma+j} \sum_{\kappa=0}^{\infty} \beta_{\kappa}^{n} (\sigma-j)^{2\kappa} h^{2\kappa} \\ &= \sum_{j=0}^{2\sigma} \binom{2\sigma}{j} (-1)^{\sigma+j} \left(\sum_{\kappa=0}^{\sigma} + \sum_{\kappa=\sigma+1}^{\infty} \beta_{\kappa}^{n} (\sigma-j)^{2\kappa} h^{2\kappa} \right). \end{aligned} \tag{4.20}$$

Since

$$\sum_{j=0}^{2\sigma} (-1)^j \binom{2\sigma}{j} j^{\nu} = 0 \qquad \text{for} \quad \nu = 0, ..., 2\sigma - 1$$

and

$$\sum_{j=0}^{2\sigma} (-1)^j \binom{2\sigma}{j} j^{2\sigma} = (2\sigma)!,$$

it follows from (4.20) that

$$M_{2\sigma}(b_{n,k}) = (-1)^{\sigma} \beta_{\sigma}^{n} (2\sigma)!\, h^{2\sigma} + \sum_{j=\sigma}^{2\sigma} \binom{2\sigma}{j} (-1)^{\sigma+j} \sum_{\kappa=\sigma+1}^{\infty} \beta_{\kappa}^{n} (\sigma-j)^{2\kappa} h^{2\kappa}. \tag{4.21}$$

It remains to show that

$$\sum_{\kappa=\sigma+1}^{\infty} \beta_{\kappa}^{n} (\sigma-j)^{2\kappa} h^{2\kappa} = O(n^{\sigma+1}h^{2\sigma+2}) \qquad \text{as} \quad nh^2 \to 0. \tag{4.22}$$

By (4.18),

$$\begin{aligned} \left| \sum_{\kappa=\sigma+1}^{\infty} \beta_{\kappa}^{n} (\sigma-j)^{2\kappa} h^{2\kappa} \right| &\leqslant \sum_{\kappa=\sigma+1}^{\infty} \frac{(\sigma-j)^{2\kappa}}{\kappa!\, 2^{\kappa}} (nh^2)^{\kappa} \\ &= (nh^2)^{\sigma+1} \sum_{\kappa=\sigma+1}^{\infty} \frac{(\sigma-j)^{2\kappa}}{\kappa!\, 2^{\kappa}} (nh^2)^{\kappa-\sigma-1} \\ &\leqslant e^{(\sigma-j)^2/2} (nh^2)^{\sigma+1} \qquad \text{if} \quad nh^2 < 1. \end{aligned}$$

This proves (4.22) and, hence, (4.19) follows from (4.21). ∎

Proof of Theorem 2.1. Suppose n is fixed and $s < n$. By (4.19) we have

$$\lim_{h \to 0} \frac{1}{h^{2s}} M_{2s}(b_{n,k}) = (-1)^s (2s)!\, \beta_s^n.$$

The result now follows immediately from Theorem 3.1. ∎

Proof of Theorem 2.2. By (4.19) and (4.13)

$$M_{2s}(b_{n,k}) = (-1)^s (2s)! \left\{ \frac{(-1)^s n^s}{(4!)^s s!} + \text{a polynomial in } n \text{ of degree } < s \right\} h^{2s} + O(n^{s+1} h^{2s+2}).$$

Hence

$$\lim_{nh \to 0} \frac{1}{(nh^2)^s} M_{2s}(b_{n,k}) = \frac{(2s)!}{(4!)^s s!},$$

and Theorem 2.2 follows from Theorem 3.1. ∎

The proof of Theorem 2.3 is the same as that of Theorem 2.2.

5. Trigonometric Moments of Trigonometric B-Spline Kernels

The asymptotic estimate for the de la Vallée Poussin–Schoenberg operators $T_{m,k}$ depends on the estimate for the corresponding trigonometric B-spline kernels $\tau_{m,k}$. For a fixed degree m an estimate for $\tau_{m,k}$ as $h := 2\pi/k \to 0$ can be obtained by the same method as for the periodic polynomial B-spline kernels. We shall omit the details which are slight modifications of those given in Section 3. However, the analysis of the asymptotic estimate for $\tau_{m,k}$ as $m \to \infty$ and $mh \to \alpha \in (0, \pi]$ requires another approach. To this end we consider a more general trigonometric B-spline p_n, $n \in \mathbf{N}_0$, which is a 2π-periodic function defined by

$$p_n(x) := (-i)^n e^{-inx/2} M_n(e^{ix}), \qquad x \in [0, 2\pi), \tag{5.1}$$

where M_n is the nth degree complex B-spline on the unit circle (see [16, 17]). The function p_n is a real-valued function supported on $[0, (n+1)h]$, $h := 2\pi/k$, and it can be expanded in the form

$$p_n(x) = \sum_{v \in \mathbf{Z}} t_v e^{i(v - n/2)(x - (n+1)h/2)}, \qquad x \in [0, 2\pi), \tag{5.2}$$

where

$$t_\nu := \begin{cases} \dfrac{2^n}{k} \displaystyle\prod_{\substack{j=0 \\ \neq \nu}}^{n} \dfrac{\sin(\nu - j)\, h/2}{(\nu - j)}, & 0 \leqslant \nu \leqslant n \\ \dfrac{2^n}{\pi} \displaystyle\prod_{j=0}^{n} \dfrac{\sin(\nu - j)\, h/2}{(\nu - j)}, & \text{otherwise.} \end{cases}$$

For $n = 2m$

$$p_{2m}(x + (2m+1)\, h/2)/t_m = \tau_{m,k}(x), \qquad x \in \mathbf{R}, \tag{5.3}$$

the trigonometric B-spline kernel defined by (1.4) and (1.5). The sequence p_n, satisfies the recurrence relation

$$np_n(x) = 2 \sin \tfrac{1}{2} x p_{n-1}(x) + 2 \sin \tfrac{1}{2}((n+1)\, h - x)\, p_{n-1}(x - h), \qquad n \in \mathbf{N} \tag{5.4}$$

(see [5]). Applying (5.4) twice with $n = 2m$ followed by $n = 2m - 1$, a straightforward computation leads to the following relation for $\tau_{m,k}$.

LEMMA 5.1. *For* $m \in \mathbf{N}$

$$\begin{aligned} \frac{2(2m-1) \sin^2 \frac{1}{2} mh}{m} \tau_{m,k}(x) = {} & \sin^2 \tfrac{1}{2}(x + (2m+1)\, h/2)\, \tau_{m-1,k}(x + h) \\ & + \sin^2 \tfrac{1}{2}((2m+1)\, h/2 - x)\, \tau_{m-1,k}(x - h) \\ & + (\cos x \cos h/2 - \cos mh)\, \tau_{m-1,k}(x). \end{aligned} \tag{5.5}$$

The next lemma gives a recurrence relation for the moments of the trigonometric B-spline kernels.

LEMMA 5.2. *Let* $m, \sigma \in \mathbf{N}$. *If* $M_{2j}(\tau_{m,k}) = O(1/m^j)$ *as* $m \to \infty$ *and* $mh \to \alpha \in (0, \pi]$ *for* $j \leqslant \sigma$; *then*

$$\begin{aligned} M_{2(\sigma+1)}(\tau_{m-1,k}) = {} & \frac{A_m}{G_m} M_{2\sigma}(\tau_{m,k}) + \frac{(B_{m,\sigma} - C_{m,\sigma})}{G_m} M_{2\sigma}(\tau_{m-1,k}) \\ & - \frac{D_{m,\sigma}}{G_m} M_{2(\sigma-1)}(\tau_{m-1,k}) + O\left(\frac{1}{m^{\sigma+2}}\right), \end{aligned} \tag{5.6}$$

where

$$A_m = \frac{8(2m-1)}{m} \sin^2 \tfrac{1}{2}mh \tag{5.7}$$

$$B_{m,\sigma} = 16\sigma \cos^{2\sigma-1} \tfrac{1}{2}h \sin \tfrac{1}{2}h \sin \tfrac{1}{2}(2m-1)\,h \tag{5.8}$$

$$C_{m,\sigma} = 8 \cos^{2\sigma} \tfrac{1}{2}h \sin^2 \tfrac{1}{4}(2m-1)\,h + 4(\cos h/2 - \cos mh) \tag{5.9}$$

$$D_{m,\sigma} = 32 \binom{2\sigma}{2} \sin^2 \tfrac{1}{2}h \cos^{2\sigma-2} \tfrac{1}{2}h \sin^2 \tfrac{1}{4}(2m-1)\,h \tag{5.10}$$

$$G_m = 2 \cos \tfrac{1}{2}(2m-1)\,h \cos^{2\sigma} \tfrac{1}{2}h - 2 \cos \tfrac{1}{2}h. \tag{5.11}$$

Proof. Multiplying (5.5) by $4(2 \sin \frac{1}{2}x)^{2\sigma}$ and integrating over the interval $[-\pi, \pi]$, gives

$$A_m M_{2\sigma}(\tau_{m,k}) = I_1 + I_2 + 4(\cos h/2 - \cos mh)\, M_{2\sigma}(\tau_{m-1,k}) \\ -2 \cos \tfrac{1}{2}h M_{2(\sigma+1)}(\tau_{m-1,k}), \tag{5.12}$$

where

$$I_1 := \frac{1}{2\pi} \int_{-\pi}^{\pi} 4 \sin^2 \tfrac{1}{2}(x + (2m-1)\,h/2)(2 \sin \tfrac{1}{2}(x-h))^{2\sigma}\, \tau_{m-1,k}(x)\, dx$$

$$I_2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} 4 \sin^2 \tfrac{1}{2}((2m-1)\,h/2 - x)(2 \sin \tfrac{1}{2}(x+h))^{2\sigma}\, \tau_{m-1,k}(x)\, dx.$$

Expanding $\sin^2 \frac{1}{2}(x + (2m-1)\,h/2)$ and $(2 \sin \frac{1}{2}(x-h))^{2\sigma}$ in powers of $2 \sin \frac{1}{2}x$ and taking into account that $\tau_{m,k}$ is even and $M_{2j}(\tau_{m,k}) = O(1/m^j)$, a straightforward calculation gives

$$\begin{aligned} I_1 &= 4 \sin^2 \tfrac{1}{4}(2m-1)\,h \cos^{2\sigma} \tfrac{1}{2}h M_{2\sigma}(\tau_{m-1,k}) \\ &\quad + 16 \binom{2\sigma}{2} \sin^2(2m-1)\,\tfrac{1}{4}\,h \sin^2 \tfrac{1}{2}h \cos^{2(\sigma-1)} \tfrac{1}{2}h M_{2(\sigma-1)}(\tau_{m-1,k}) \\ &\quad + \cos(2m-1)\,\tfrac{1}{2}h \cos^{2\sigma} \tfrac{1}{2}h M_{2(\sigma+1)}(\tau_{m-1,k}) \\ &\quad - 4 \binom{2\sigma}{1} \sin \tfrac{1}{2}(2m-1)\,h \sin \tfrac{1}{2}h \cos^{2\sigma-1} \tfrac{1}{2}h M_{2\sigma}(\tau_{m-1,k}) + O\left(\frac{1}{m^{\sigma+2}}\right). \end{aligned}$$

Since k_n is even, it is clear that $I_2 = I_1$. Substituting the estimates for I_1 and I_2 into (5.12) leads to (5.6). ∎

LEMMA 5.3. *Let $m, \sigma \in \mathbf{N}$. As $m \to \infty$ and $mh \to \alpha \in (0, \pi]$,*

$$G_m = -4\sin^2 \tfrac{1}{2}mh + O\left(\frac{1}{m}\right) \tag{5.13}$$

$$\frac{A_m}{G_m} = -4 + O\left(\frac{1}{m}\right) \tag{5.14}$$

$$\frac{(m-1)\,B_{m,\sigma}}{G_m} = -8\sigma m \sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh + O\left(\frac{1}{m}\right) \tag{5.15}$$

$$\frac{(m-1)^2\,D_{m,\sigma}}{G_m} = -8\binom{2\sigma}{2}(m \sin \tfrac{1}{2}h)^2 + O\left(\frac{1}{m}\right) \tag{5.16}$$

$$\frac{(m-1)(A_m - C_{m,\sigma})}{G_m} = 2(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh) + O\left(\frac{1}{m}\right). \tag{5.17}$$

Proof. From (5.11)

$$\begin{aligned} G_m &= 2(1 - 2\sin^2 \tfrac{1}{4}(2m-1)\,h)\cos^{2\sigma} \tfrac{1}{2}h - 2\cos \tfrac{1}{2}h \\ &= -4\sin^2 \tfrac{1}{4}(2m-1)\,h + O(h^2) \\ &= -4\sin^2 \tfrac{1}{2}mh + O(mh^2) \end{aligned}$$

as $m \to \infty$ and $mh \to \alpha \in (0, \pi]$.

The estimate (5.14) follows from (5.13) and

$$A_m = 16\sin^2 \tfrac{1}{2}mh + O(mh^2) \tag{5.18}$$

which is obtained directly from (5.7).

By (5.8) and (5.13)

$$\begin{aligned} \frac{(m-1)\,B_{m,\sigma}}{G_m} &= \frac{-4(m-1)\,\sigma\cos^{2\sigma-1} \tfrac{1}{2}h \sin \tfrac{1}{2}h \sin \tfrac{1}{2}(2m-1)\,h}{\sin^2 \tfrac{1}{2}mh} + O\left(\frac{1}{m}\right) \\ &= -8\sigma \sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh + O\left(\frac{1}{m}\right). \end{aligned}$$

The estimate (5.16) is obtained in the same manner.

To prove (5.17), we first express

$$\begin{aligned} (m-1)(A_m - C_{m,\sigma}) = \frac{(m-1)}{m^2}\{&8m(2m-1)\sin^2 \tfrac{1}{2}mh - 8m^2 \sin^2 \tfrac{1}{4}\cos^{2\sigma} \tfrac{1}{2}h \\ &-4m^2(\cos \tfrac{1}{2}h - \cos mh)\}, \end{aligned}$$

using (5.7) and (5.9). Expanding $\sin^2 \frac{1}{4}(2m-1)h$ in terms of $\sin \frac{1}{2}mh$, $\cos \frac{1}{2}mh$, $\sin \frac{1}{4}h$, and $\cos \frac{1}{4}h$, a straightforward simplification leads to

$$(m-1)(A_m - C_{m,\sigma}) = -(8\sin^2 \tfrac{1}{2}mh)(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh) + O\left(\frac{1}{m}\right). \tag{5.19}$$

The estimate (5.17) now follows from (5.19) and (5.13). ∎

LEMMA 5.4. *Let $m, \sigma \in \mathbf{N}$, and suppose that*

$$m^\sigma M_{2\sigma}(\tau_{m,k}) = \frac{(2\sigma)!}{\sigma!}(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh)^\sigma + O\left(\frac{1}{m}\right) \tag{5.20}$$

as $m \to \infty$ and $mh \to \alpha \in (0, \pi]$. Then

$$\begin{aligned}(m-1)^{\sigma+1} &\{M_{2\sigma}(\tau_{m,k}) - M_{2\sigma}(\tau_{m-1,k})\}\\ &= \frac{-(2\sigma)!}{(\sigma-1)!}(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh)^{\sigma-1}\\ &\quad \times\left(1 - \frac{(m\sin \frac{1}{2}h)^2}{\sin \frac{1}{2}mh \sin \frac{1}{2}(m-1)h}\right) + O\left(\frac{1}{m}\right).\end{aligned} \tag{5.21}$$

Proof. Writing

$$\begin{aligned}(m-1)^{\sigma+1} &\{M_{2\sigma}(\tau_{m,k}) - M_{2\sigma}(\tau_{m-1,k})\}\\ &= (m-1)\{(m-1)^\sigma M_{2\sigma}(\tau_{m,k}) - (m-1)^\sigma M_{2\sigma}(\tau_{m-1,k})\}\end{aligned}$$

and expanding $(m-1)^\sigma$, the coefficient of $M_{2\sigma}(\tau_{m,k})$ on the right of the equation, in powers of m, leads to

$$\begin{aligned}(m-1)^{\sigma+1} &\{M_{2\sigma}(\tau_{m,k}) - M_{2\sigma}(\tau_{m-1,k})\}\\ &= (m-1)\{m^\sigma M_{2\sigma}(\tau_{m,k}) - (m-1)^\sigma M_{2\sigma}(\tau_{m-1,k})\}\\ &\quad - \sigma m^\sigma M_{2\sigma}(\tau_{m,k}) + O\left(\frac{1}{m}\right)\\ &= \frac{(m-1)(2\sigma)!}{\sigma!}\{(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh)^\sigma\\ &\quad - (1 - (m-1)\sin \tfrac{1}{2}h \cot \tfrac{1}{2}(m-1)h)^\sigma\}\\ &\quad - \frac{(2\sigma)!}{(\sigma-1)!}(1 - m\sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh)^\sigma + O\left(\frac{1}{m}\right).\end{aligned} \tag{5.22}$$

The last equation is obtained by applying the assumption (5.20). The expression

$$(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma}-(1-(m-1)\sin\tfrac{1}{2}h\cot\tfrac{1}{2}(m-1)h)^{\sigma}$$

in (5.22) can be written in the form

$$\begin{aligned}&\{(m-1)\sin\tfrac{1}{2}h\cot\tfrac{1}{2}(m-1)h-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh\}\\&\quad\times\sum_{j=0}^{\sigma-1}(1-(m-1)\sin\tfrac{1}{2}h\cot\tfrac{1}{2}(m-1)h)^{j}\\&\quad\times(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma-1-j}\\&=\left\{\frac{m\sin^2\tfrac{1}{2}h}{\sin\tfrac{1}{2}mh\sin\tfrac{1}{2}(m-1)h}-\sin\tfrac{1}{2}h\cot\tfrac{1}{2}(m-1)h\right\}\\&\quad\times\sigma(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma-1}+O\left(\frac{1}{m}\right).\end{aligned}$$

With this estimate, (5.22) becomes

$$\begin{aligned}&(m-1)^{\sigma+1}\{M_{2\sigma}(\tau_{m,k})-M_{2\sigma}(\tau_{m-1,k})\}\\&\quad=\frac{(2\sigma)!}{(\sigma-1)!}(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma-1}\frac{(m\sin\tfrac{1}{2}h)^2}{\sin\tfrac{1}{2}mh\sin\tfrac{1}{2}(m-1)h}\\&\quad-\frac{(2\sigma)!}{(\sigma-1)!}(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma-1}m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}(m-1)h\\&\quad-\frac{(2\sigma)!}{(\sigma-1)!}(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{\sigma}+O\left(\frac{1}{m}\right)\end{aligned}$$

which simplifies to (5.21). ∎

THEOREM 5.1. *For $m, j\in\mathbf{N}$,*

$$m^{j}M_{2j}(\tau_{m,k})=\frac{(2j)!}{j!}(1-m\sin\tfrac{1}{2}h\cot\tfrac{1}{2}mh)^{j}+O\left(\frac{1}{m}\right)\tag{5.23}$$

as $m\to\infty$ and $mh\to\alpha\in(0,\pi]$.

Proof. We shall establish the result by induction on j. When $j=1$, (2.3) gives

$$mM_2(\tau_{m,k})=2m(1-\hat{\tau}_1),$$

and a straightforward simplification using (1.5) leads to

$$mM_2(\tau_{m,k}) = 2(1 - m \sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh) + O\left(\frac{1}{m}\right). \tag{5.24}$$

Therefore (5.23) holds for $j = 1$.

Suppose that (5.23) holds for $j \leqslant \sigma$. Then by (5.6),

$$\begin{aligned}(m-1)^{\sigma+1} & M_{2(\sigma+1)}(\tau_{m-1,k}) \\ &= \frac{(m-1)^{\sigma+1} A_m}{G_m} (M_{2\sigma}(\tau_{m,k}) - M_{2\sigma}(\tau_{m-1,k})) \\ &+ \frac{(m-1)^{\sigma+1} B_{m,\sigma}}{G_m} M_{2\sigma}(\tau_{m-1,k}) \\ &+ \frac{(m-1)^{\sigma+1} (A_m - C_{m,\sigma})}{G_m} M_{2\sigma}(\tau_{m-1,k}) \\ &- \frac{(m-1)^{\sigma+1} D_{m,\sigma}}{G_m} M_{2(\sigma-1)}(\tau_{m-1,k}) + O\left(\frac{1}{m}\right).\end{aligned}$$

Applying the results of Lemmas 5.3 and 5.4 and the inductive hypothesis to each term on the right-hand side, the above equation simplifies to

$$\begin{aligned}(m-1)^{\sigma+1} & M_{2(\sigma+1)}(\tau_{m-1,k}) \\ &= \frac{(2\sigma+2)!}{(\sigma+1)!} (1 - m \sin \tfrac{1}{2}h \cot \tfrac{1}{2}mh)^{\sigma+1} + O\left(\frac{1}{m}\right).\end{aligned}$$

The result now follows by induction. ∎

Proof of Theorem 2.5. It follows from (5.23) that for $j \in \mathbf{N}$

$$\lim_{m \to \infty} m^j M_{2j}(\tau_{m,k}) = \frac{(2j)!}{j!} \left(1 - \frac{\alpha}{2} \cot \frac{\alpha}{2}\right)^j,$$

where the limit is taken as $m \to \infty$ and $mh \to \alpha \in (0, \pi]$. Theorem 2.5 now follows immediately from Theorem 3.1. ∎

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